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Analysis of a beam-column system under varying axial forces of elliptic type: the exact solution of Lamé's equation

Alex Elías-Zúñiga

Departamento de Ingeniería Mecánica, Instituto Tecnológico y de Estudios Superiores de Monterrey, E. Garza Sada 2501 Sur, C.P. 64849, Monterrey, NL, Mexico

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Abstract

We investigate the dynamical response of a beam-column system with hinged ends subjected to an axial pulsating force of elliptic type. It is shown that the resulting equation is of the form

$$\frac{d^2y_1}{d\tau^2} + y_1[d_1 + d_2cn^2(\tau,k^2)] = 0,$$

which is the well-known Lamé equation [Higher Transcendental Functions, Bateman Manuscript Project, edited by McGraw-Hill, New York, vol. 3]. In this paper, we obtain the general exact solution of this equation that reveals stable behavior of the beam-column system if the assigned initial conditions are of the form $y_1(0) = y_{10}$ and $\dot{y}_1(0) = 0$. It is also found that at a certain value of the modulus of the elliptic force, the lateral vibrational frequency is independent of the material properties of the beam-column system.

Keywords: Jacobi elliptic functions; Lateral vibration; Euler load; Elastic stability; Buckling

1. Introduction

In this paper we investigate the main physical characteristics of an elastic hinged beam-column system subjected to a pulsating load of elliptic type. It is well-known that an elastic beam-column with hinged ends can be put into stable equilibrium by applying pulsating loads at the proper driving frequency (Lubkin and Stoker, 1943). If the pulsating load is of the sine or cosine type, then the resulting governing equation of motion reduces to the well-known Mathieu equation whose exact solution is not known and hence, numerical schemes or perturbation techniques are used to obtain an approximate solution (Klötter and Kotowski, 1943; Stoker, 1950; Porter, 1962; Rand, 1969; Nayfeh and Mook, 1973).

However, when the pulsating load is given as a function of Jacobian elliptic functions, the resulting Lamé equation has an exact solution. In this paper, we study the main characteristics of the behavior of a hinged beam-column under the action of this type of pulsating loads.

E-mail address: aelias@itesm.mx (A. Elías-Zúñiga).

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2. Problem formulation

An elastic uniform beam-column with hinged ends and length L subjected to the action of a compressive varying axial force is shown in Fig. 1. This force is described by

$$f(t) = P + f^*(t), \tag{2.1}$$

where P is a compressive stationary force and $f^*(t)$ is a periodically varying force with driving frequency Ω.

It is known (Stoker, 1950) that the differential equation that describes the lateral deflection $w(x,t)$ of the beam-column system is given by

$$EI\frac{\partial^4 w}{\partial x^4} + f(t)\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0. \tag{2.2}$$

In this equation E is the Young's modulus, I is the area moment of inertia of the cross section, t is the running time, and m represents the mass of the column per unit length. The boundary conditions are those corresponding to the case of a hinged beam-column i.e., the lateral deflection $w(x,t)$ and the bending moment $M = EI(\partial^2 w/\partial x^2)$ are both zero at $x = 0$ and $x = L$. Therefore, the boundary conditions for (2.2) at $x = 0$ and $x = L$ for all t are:

$$w = \frac{\partial^2 w}{\partial x^2} = 0. \tag{2.3}$$

These boundary conditions can be satisfied by taking for the lateral deflection $w(x,t)$ a solution in the form of a Fourier series:

$$w(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin\frac{n\pi x}{L}. \tag{2.4}$$

Substitution of (2.4) into (2.2) yields

$$\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left[EI\left(\frac{n\pi}{L}\right)^4 y_n(t) - f(t)\left(\frac{n\pi}{L}\right)^2 y_n(t) + m\ddot{y}_n(t)\right] = 0. \tag{2.5}$$

If only the first mode of vibration of the beam-column system is considered, then (2.5) reduces to

$$\sin\frac{\pi x}{L}\left[EI\left(\frac{\pi}{L}\right)^4 y_1(t) - f(t)\left(\frac{\pi}{L}\right)^2 y_1(t) + m\ddot{y}_1(t)\right] = 0. \tag{2.6}$$

To obtain a non-trivial solution of Eq. (2.6), we must have

$$\ddot{y}_1(t) + y_1(t)\left[\frac{EI}{m}\left(\frac{\pi}{L}\right)^4 - \frac{1}{m}\left(\frac{\pi}{L}\right)^2 f(t)\right] = 0, \tag{2.7}$$

which represents the governing equation of motion of the beam-column system with hinged ends. Note that the frequency of the lateral vibration of the beam-column without axial load is given by

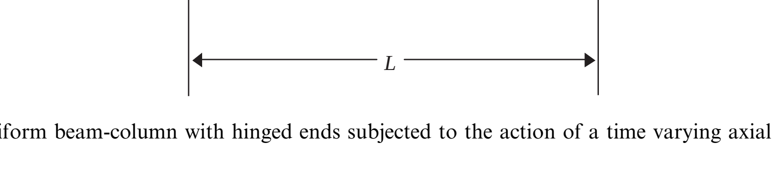

Fig. 1. Elastic uniform beam-column with hinged ends subjected to the action of a time varying axial compressive force.

$$\omega_0^2 = \frac{\pi^4 EI}{mL^4}. \tag{2.8}$$

Introducing the dimensionless time $\tau = \Omega t$, Eq. (2.7) can be written as

$$\frac{d^2 y_1(\tau)}{d\tau^2} + y_1(\tau)\left[\frac{EI}{m\Omega^2}\left(\frac{\pi}{L}\right)^4 - \frac{1}{m\Omega^2}\left(\frac{\pi}{L}\right)^2 f(\tau)\right] = 0. \tag{2.9}$$

Substituting (2.1) into (2.9) gives

$$\frac{d^2 y_1(\tau)}{d\tau^2} + y_1(\tau)[d_1 + d_2^* f^*(\tau)] = 0, \tag{2.10}$$

where

$$d_1 = \frac{\pi^2}{mL^2\Omega^2}\left[EI\left(\frac{\pi}{L}\right)^2 - P\right], \tag{2.11}$$

and

$$d_2^* = -\left(\frac{\pi}{L}\right)^2 \frac{1}{m\Omega^2}. \tag{2.12}$$

Recalling that the Euler load for this beam-column is given by $P_e = \pi^2 EI/L^2$ and introducing Eq. (2.8), the parameters d_1 and d_2^* can be written as

$$d_1 = \frac{\omega_0^2}{\Omega^2}\left(1 - \frac{P}{P_e}\right), \tag{2.13}$$

$$d_2^* = -\frac{\omega_0^2}{\Omega^2}\frac{1}{P_e}. \tag{2.14}$$

If $f^*(\tau)$ is equal to zero, then (2.10) reduces to the case of lateral vibration of a beam-column with static load P. In this case, Eq. (2.13) represents the dimensionless frequency of lateral vibrations. Note that if $P < P_e$ then the beam-column system has a simple harmonic response that remains stable. When P reaches the value of the Euler load, the lateral frequency becomes zero. Under this load there will no longer be any vibration, and the beam-column is in equilibrium in a slightly deflected form. The case for which $P > P_e$ produces an exponential type solution of Eq. (2.10) that increases with time. Hence the motion of the beam-column system is unstable.

Next, we shall study the solution of Eq. (2.10) and the response of the beam-column system when $f^*(\tau)$ is an axial force of elliptic type.

3. Exact solution of Lamé's equation

Assuming certain periodic functions for $f^*(t)$, we may show that Eq. (2.10) can be solved exactly. These functions that provide exact solutions for (2.10) are known as Jacobian elliptic functions; i.e. the Jacobian elliptic function $cn(\tau, k^2)$ and $sn(\tau, k^2)$ that have a period in τ equal to $4K(k^2)$, where $K(k^2)$ is the complete elliptic integral of the first kind for the modulus k (Byrd and Friedman, 1953). Also, the Jacobian elliptic function $dn(\tau, k^2)$ has a period in τ equal to $2K(k^2)$. Now, let us investigate the behavior of the beam-column system assuming a periodic driving function $f^*(t)$ of the form

$$f^*(t) = a\,cn^2(\Omega t, k^2), \tag{3.1}$$

where a is the magnitude of the driving force. To make the Jacobian elliptic function $cn(\Omega t, k^2)$ to be of period 4π (Hsu, 1974), the relation between the modulus k and the driving frequency Ω has to be given by

$$\Omega = \frac{K(k^2)}{\pi}. \tag{3.2}$$

Notice that when $k \to 1$, the driving frequency $\Omega \to \infty$; and if $k \to 0$ then $\Omega \to 1/2$.

Substituting (3.1) into (2.10), and using the dimensionless time τ, gives

$$\frac{d^2 y_1}{d\tau^2} + y_1[d_1 + d_2 cn^2(\tau, k^2)] = 0, \tag{3.3}$$

where

$$d_2 = -\frac{\omega_0^2}{\Omega^2}\frac{a}{P_e}. \tag{3.4}$$

The initial conditions are assumed to be

$$y_1(0) = y_{10}, \quad \dot{y}_1(0) = \dot{y}_{10}. \tag{3.5}$$

By defining the relation

$$y = \frac{y_1}{y_{10}}, \tag{3.6}$$

then (3.3) can be written as

$$\frac{d^2 y}{d\tau^2} + y[d_1 + d_2 cn^2(\tau, k^2)] = 0, \tag{3.7}$$

with appropriate initial conditions. Eq. (3.7) is a special form of the well-known Lamé equation. This is a second-order linear ordinary differential equation whose exact solution must have two linearly independent solutions. We assumed here that one of these linearly independent solution is of the form:

$$y = cn(\tau, k^2). \tag{3.8}$$

Substitution of Eq. (3.8) into Eq. (3.7), gives

$$cn(\tau, k^2)[d_1 + 2k^2 - 1] + cn^3(\tau, k^2)[d_2 - 2k^2] = 0. \tag{3.9}$$

Eq. (3.9) holds for all τ if and only if each coefficient vanishes, i.e. provided that

$$d_1 = 1 - 2k^2, \tag{3.10}$$

$$d_2 = 2k^2. \tag{3.11}$$

The exact solution (3.8) can be used to find the second linearly independent solution (see O'Neil, 1991, pp. 113–115). The idea is to look for a second solution of the form

$$y^*(\tau) = v(\tau)y(\tau), \tag{3.12}$$

in which v is a nonconstant function of τ given by

$$v(\tau) = \int \frac{1}{y(\tau)^2}\, d\tau = \int \frac{1}{cn(\tau, k^2)^2}\, d\tau. \tag{3.13}$$

Note that (3.13) has an integral solution (Byrd and Friedman, 1953) given by

$$v(\tau) = \frac{1}{(k^2-1)}\left(k^2\left[\tau\left(\frac{k^2-1}{k^2}\right) + \frac{E(\psi,k^2)\left\{cn(\tau,k^2)^2 + \frac{1}{k^2} - 1\right\}}{dn(\tau,k^2)\sqrt{1-k^2 sn(\tau,k^2)^2}}\right] - \frac{dn(\tau,k^2)sn(\tau,k^2)}{cn(\tau,k^2)}\right), \tag{3.14}$$

where $E(\psi,k^2)$ represents the incomplete elliptic intergral of the second kind and $\psi = am(\tau,k^2)$ is called the amplitude. Substitution of Eq. (3.14) into Eq. (3.12) provides the second linearly independent solution

$$y^*(\tau) = \frac{cn(\tau,k^2)}{(k^2-1)}\left(k^2\left[\tau\left(\frac{k^2-1}{k^2}\right) + \frac{E(\psi,k^2)\left\{cn(\tau,k^2)^2 + \frac{1}{k^2} - 1\right\}}{dn(\tau,k^2)\sqrt{1-k^2 sn(\tau,k^2)^2}}\right] - \frac{dn(\tau,k^2)sn(\tau,k^2)}{cn(\tau,k^2)}\right). \tag{3.15}$$

Thus, the general exact solution of Lamé's equation becomes [1]

$$y(\tau) = cn(\tau,k^2)\left(C_1 + \frac{C_2}{(k^2-1)}\left[k^2\left\{\tau\left(\frac{k^2-1}{k^2}\right) + \frac{E(\psi,k^2)\left[cn(\tau,k^2)^2 + \frac{1}{k^2} - 1\right]}{dn(\tau,k^2)\sqrt{1-k^2 sn(\tau,k^2)^2}}\right\} - \frac{dn(\tau,k^2)sn(\tau,k^2)}{cn(\tau,k^2)}\right]\right), \tag{3.16}$$

where C_1 and C_2 are integration constants that can be determined from the assigned initial conditions. For instance, if the initial conditions are assumed to be given by $y(0) = 1$ and $\dot{y}(0) = \dot{y}_0$ then $C_1 = 1$ and $C_2 = \dot{y}_0$ and hence our solution (3.16) grows without bounds as time τ increases and hence, the beam-column system has unstable behavior no matter what values are chosen for a, P, and k that satisfy relations (3.10) and (3.11). But if we choose the initial conditions to be given by $y(0) = 1$ and $\dot{y}(0) = 0$, the integration constants become $C_1 = 1$ and $C_2 = 0$ and hence the exact solution of Lamé's equation becomes bounded for all time τ. Recalling Eqs. (3.10) and (3.11) and if the value of the modulus k, or d_2, is given then we can find d_1 and d_2, or k and d_1, and then the values of a and P, from (2.13) and (3.4), for stable beam-column behaviour. Note that for all values of d_1 and d_2 obtained from Eqs. (3.10) and (3.11), the solution (3.8) is bounded and stable. Fig. 2 shows the variation of d_1 and d_2 with Ω where it can be seen that these curves intersect at the value of $\Omega = 0.5365$ rad/s for which $d_1 = d_2 = 1/2$.

Fig. 3 shows the plots of d_1 and d_2 versus the modulus of the Jacobian elliptic function k. Note that d_1 and d_2 intersect at $k = 1/2$.

Now, using (3.10) and (3.11) and recalling Eqs. (2.13) and (3.4), it is possible to obtain the relation between the Euler load P_e and the axial forces a and P:

$$\frac{P_e}{(a+P)} = \frac{\left(\frac{\omega_0}{\Omega}\right)^2}{\left(\frac{\omega_0}{\Omega}\right)^2 - 1}. \tag{3.17}$$

Notice from Eq. (3.17) that the driving force a is in tension as long as $0 < \omega_0/\Omega \leqslant 1$. It is also seen in Fig. 4 that when $0 < \omega_0/\Omega \leqslant \sqrt{2}/2$, $|a+P| > P_e$. In this case, small oscillations of the beam-column system in

[1] We know from (Magnus and Winkler, 1979) that Lamé's equation (3.7) can be transformed into an equation of Ince's type by substituting $\tau = am(u,k^2)$ and thus, our exact solution derived here for Lamé's equation also holds for Ince's equation and for all other equations that can be cast into an equation of this type. We shall not elaborate any further the details of the exact solution of Ince's equation and leave this for publication elsewhere.

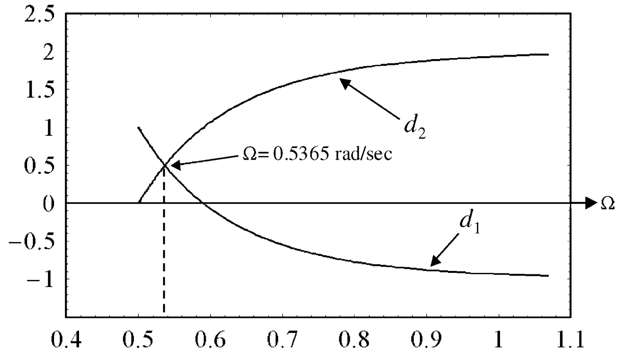

Fig. 2. Plot of the d_1 and d_2 versus the driving frequency Ω. Notice that these curves intersect at the value of $\Omega = 0.5365$ rad/s.

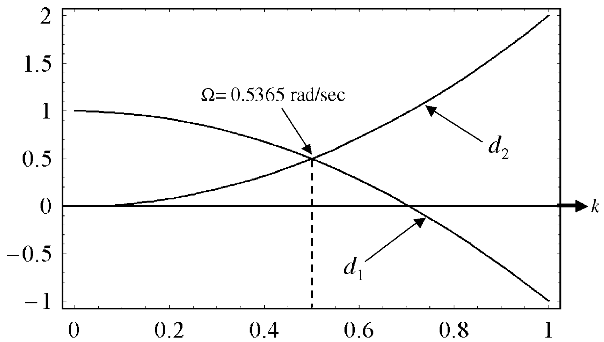

Fig. 3. Plot of d_1 and d_2 versus the modulus k.

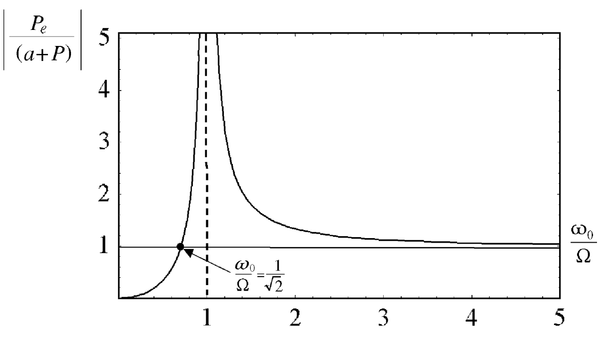

Fig. 4. Frequency response diagram.

the neighborhood of its undeflected position are stable. When the ratio ω_0/Ω approaches the resonance condition of the beam-column system, the ratio $P_e/(a+P)$ becomes indeterminately large. This corresponds to the case for which the magnitude of the tension driving force a becomes close to the value of the compressive stationary force P. For any further increase in the frequency ratio ω_0/Ω, the magnitude of the ratio $P_e/(a+P)$ approaches asymptotically to the value of one. Thus, the magnitude of $(a+P)$ that is acting on the beam becomes close to the magnitude of the Euler load P_e.

Since the intersection points in Figs. 2 and 3 correspond to the value of $\omega_0/\Omega = 1/\sqrt{2}$ at which $|a| = P_e$ in Fig. 4, the value of the lateral vibrational frequency of the beam-column system is found to be $\omega_0 = 0.3794$ rad/s in which Eq. (3.2) has been used. Note that the value of ω_0 is independent of the elastic material properties of the beam-column system. Therefore, the following can be concluded:

The value of $\omega_0 = 0.3794$ *rad/s represents a universal constant frequency of the lateral vibration of a hinged beam-column system, valid for any elastic material for which the Euler load* P_e *and the absolute value of the pulsating load a are equal.* Substitution of this value into Eq. (2.8), gives the relation for which the pulsating and Euler loads are equal

$$P_{eu} = 0.1439\frac{mL^2}{\pi^2}. \qquad (3.18)$$

The above equation shows that the load is independent of the Young's modulus and therefore, the following can be concluded:

The universal load P_{eu}*, for which* $|a| = P_e$*, valid for any kind of elastic material and given by (*3.18*) depends only on the mass m and the length L of the beam-column system.*

3.1. Stability–instability chart

Eq. (3.7) contains only three parameters, to say k, d_1, and d_2. The parameter k represents the modulus of the jacobian elliptic function *cn* and it is related to the driving frequency Ω through Eq. (3.2). The parameter d_1 is related to the static load P while d_2 depends on the magnitude of the driving force a. It is well-known that for arbitrary values of these parameters that do not follow relations (3.10) and (3.11), the general solution of Eq. (3.7) may be stable or unstable. Since Eq. (3.7) is an equation with periodic coefficients, we may determine the stability–instability chart by using numerical integration in conjunction with Floquet theory (Rand, 2001).

Fig. 5 shows a typical stability–instability chart for the Lamés equation (3.7) for the value of $k = 1/2$. The shaded (unshaded) regions of the chart indicate values of d_1 and d_2 for which the solutions are stable (unstable). This stability–instability chart is similar to the one obtained by Greene et al. (1997) with the difference that they plotted it for the value of $k = 1/\sqrt{2}$ and for positive values of d_1 and d_2.

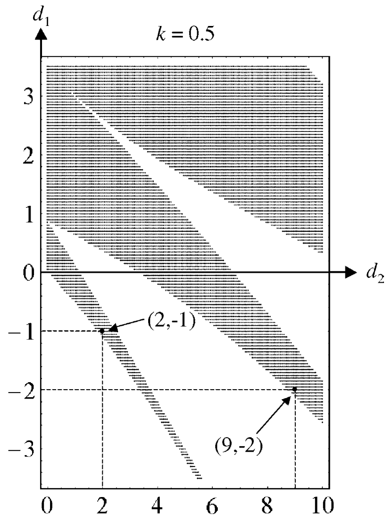

Fig. 5. Stability–instability chart for the Lamé's equation obtained from the numerical solution of Eq. (3.7) for the value of $k = 0.5$.

Recalling Eqs. (2.13) and (3.4), we may obtain a relation between the driving force a and the Euler load P_e:

$$a = -\frac{d_2}{d_1}(P_e - P). \tag{3.19}$$

We can see from Fig. 5 that in some regions of the stability–instability chart the value of the driving force a is bigger than the Euler load P_e. For instance, if we take the value of $P = 0$ and if we pick the value of $d_1 = -1$ and the value of $d_2 = 2$ and in accordance with (3.19), the driving force is twice that of the Euler load without producing lateral buckling in the beam-column system. Similar conclusion can be drawn if we take the values of $P = 5/9P_e$, $d_1 = -2$, and $d_2 = 9$. Thus, there exist set of values of d_1, d_2, and P for which $a > P_e$ without causing buckling in the beam-column system.

Finally, if the the driving forces are replaced by either

$$f^*(\tau) = asn^2(\tau, k^2) + bcn^2(\tau, k^2), \tag{3.20}$$

or

$$f^*(\tau) = asn^2(\tau, k^2) + bcn^2(\tau, k^2) + cdn^2(\tau, k^2), \tag{3.21}$$

the resulting equation of motion (2.10) also has exact solution. We shall describe this procedure in future work.

4. Conclusions

The effects of applying an axial force of elliptic type to a hinged beam-column system have been studied. It was shown that under these type of loads, Lamé's equation has a closed-form solution. Special focus was given for describing the dynamical response of the system by using the obtained closed-form solution. It was observed that when $0 < \omega_0/\Omega < 1/\sqrt{2}$, the absolute value of the magnitude of the axial loads is larger than the Euler load without producing lateral deflection on the beam-column system. At the frequency ratio value of $\omega_0/\Omega = 1/\sqrt{2}$ for which $k = 1/2$, we found that the value of the lateral vibration frequency of the beam-column system is independent of the material properties and that the absolute value of the pulsating load is equal to the Euler load.

It was also shown that for certain set of values of the parameters d_1, d_2, and P, the driving force a is bigger than the Euler load without producing lateral buckling in the beam-column system.

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